ON CERTAIN DIOPHANTINE EQUATIONS RELATED TO TRIANGULAR AND TETRAHEDRAL NUMBERS

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ABSTRACT. In this paper we give solutions of certain diophantine equations related to triangular and tetrahedral numbers and propose several problems connected with these numbers.

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1. Introduction

By a *triangular number* we call the number of the form

$$t_n = 1 + 2 + \ldots + n - 1 + n = \frac{n(n+1)}{2},$$

where n is a natural number. The number t_n can be interpreted as a number of circles necessary to build an equilateral triangle with side of length n. In analogous manner we define tetrahedral number T_n, which gives number of balls necessary to to build tetrahedron with side of length n. More explicite value of T_n is given by

$$T_n = t_1 + t_2 + \ldots + t_{n-1} + t_n = \frac{n(n+1)(n+2)}{6}.$$

Wacław Sierpiński in the booklet [4] and in the papers [3, 5, 6, 7, 8] gave many interesting results concerning the problem of solvability of diophantine equations related to triangular nad tetrahedral numbers. The aim of this paper is to give solutions of certain diophantine problems which was left as open in the booklet [4] and give some new results. Our proofs are of elementary character and we do not assume any special knowledge from number theory.

The number of possible problems which can be stated in connection with triangular and tetrahedral numbers is bounded only by imagination and with interest of researcher. This is the reason of selection of problems in this paper. We encourage the reader to solve problems mentioned in this paper and to state own problems.

2. Triangular and tetrahedral numbers as sides of right triangles

In this section we are interested in the construction of right triangles with such a property that are triangular numbers.

We start with the problem related to the construction of right triangles with legs which are triangular numbers. So, we will be interested in integers solutions of the diophantine equation

$$(1) \qquad\qquad z^2 = t_x^2 + t_y^2,$$

W. Sierpiński in [4, page 34] has shown that the above equation has infinitely many solutions in integers. However, all solutions presented by him satisfied the condition $\text{GCD}(t_x, t_y) > 1$. In other words, triple $X = t_x$, $Y = t_y$, $Z = z$ which satisfy the equation $X^2 + Y^2 = Z^2$ is not primitive solution. A. Schinzel showed that bthe set of integer solutions of the equation (1) which satisfy the condition $\text{GCD}(t_x, t_y) = 1$ is infinite. It is natural to ask the following question: does the equation (1) have parametric solutions? In other words: does the equation (1) have solutions in the ring $\mathbb{Z}[u]$?

Generally, these sort of questions is very difficult and we do not have any general theory which could by used. However, as we will see, for our particular equation it is possible to construct infinitely many polynomials $x(u), y(u), z(u) \in \mathbb{Z}[u]$ which satisfy the condition $\text{GCD}(t_{x(u)}, t_{y(u)}) = 1$.

It is clear that we can consider the equation

$$r^2 = (p^2 - 1)^2 + (q^2 - 1)^2.$$

Indeed, if p, q, r satisfied the above equation, then the triple of integers $(p-1)/2, (q-1)/2, r/8$ will be solution of (1). From this remark we can see that the quantities $p^2 - 1, q^2 - 1, r$ must be solutions of the equation of Pythagoras $Z^2 = X^2 + Y^2$. It is well known that, all solutions of this equation are of the form

$$X = 2abc, \quad Y = (a^2 - b^2)c, \quad Z = (a^2 + b^2)c,$$

where a, b, c are certain integers. Let us put $c = 1$ and consider the system of equations given by

$$p^2 = 2ab + 1, \quad q^2 = a^2 - b^2 + 1.$$

First equation of the above system will be satisfied if we put

$$a = \frac{u(ku - 2)}{2}, \quad b = k, \quad p = ku - 1.$$

We put the quantities given above into the equation $q^2 = a^2 - b^2 + 1$ and we get

$$q^2 = \frac{u^4 - 4}{4}k^2 - u^3 k + u^2 + 1 =: f(k).$$

This is Pell type equation depending on the parameter u. Let us note that $f(1) = (u(u-2)/2)^2$ and that the following identity holds

$$f\left(\frac{(u^4 - 2)k}{2} + u^2 q - u^3\right) - \left(\frac{u^2(u^4 - 4)k + 2(u^4 - 2)q - 2u^5}{4}\right)^2 = f(k) - q^2.$$

From the above we can deduce that if we define

(2)
$$\begin{cases} k_0 = 1, \quad q_0 = u(u-2)/2, \\ k_n = \frac{(u^4 - 2)k_{n-1}}{2} + u^2 q_{n-1} - u^3, \\ q_n = \frac{u^2(u^4 - 4)k_{n-1} + 2(u^4 - 2)q_{n-1} - 2u^5}{4}, \end{cases}$$

then, the polynomials $p_n(u) = k_n(u)u - 1$, $q_n(u)$, $r_n(u) = u^2(k_n(u) - 2)^2/4 + k_n(u)^2$ for $n = 1, 2, \ldots$ satisfy the equation $(p^2 - 1)^2 + (q^2 - 1)^2 = r^2$. Finally, we get that the polynomials $x_n(u) = (p_n(u) - 1)/2$, $y_n(u) = (q_n(u) - 1)/2$, $z_n(u) = r_n(u)/8$

satisfied the equation $t_x^2 + t_y^2 = z^2$. In particular, for $n = 1$ we get

$$x_1(u) = (u^5 - 2u^4 - u - 2)/2,$$

$$y_1(u) = (u^2 + 1)(u^4 - 2u^3 - u^2 + 2u - 2)/4,$$

$$z_1(u) = (u^{12} - 4u^{11} + 4u^{10} + 2u^8 - 16u^7 + 24u^6 - 7u^4 + 20u^3 + 4u^2 + 4)/32.$$

Resultant $\mathrm{Res}(t_{x_1}, t_{y_1})$ of the polynomials $t_{x_1(u)}, t_{y_1(u)}$ is equal to 2^{-58}, which means that the polynomials are co-prime. Let us note that the polynomials $x_n(2u + 1), y_n(2u + 1), z_n(2u + 1)$ belong to $\mathbb{Z}[u]$. It is possible to prove (we will not do this here) that for each positive integer n thew polynomials $t_{x_n(u)}, t_{y_n(u)}$ are co-prime. We have proved the following

Theorem 2.1. *The equation $z^2 = t_x^2 + t_y^2$ has infinitely many solutions in polynomials $x(u), y(u), z(u) \in \mathbb{Z}[u]$.*

Remark 2.2. The problem of construction of the right angle triangles which all sides are triangular numbers so the problem of the construction of integer solutions of the equation $t_x^2 + t_y^2 = t_z^2$ was posed by K. Zarankiewicz. Only one integer solution of this equation is known so far: $x = 132, y = 143, z = 164$. In connection with this problem we prove the following

Theorem 2.3. *The equation $t_x^2 + t_y^2 = t_z^2$ has infinitely many solutions in rational numbers.*

Proof. Because we are interested in rational solution of our equation, so without loss of generality we can consider the equation

$$(p^2 - 1)^2 + (q^2 - 1)^2 = (r^2 - 1)^2.$$

Let u, v be indeterminate parameters. Let us put

$$p = 2uvT - 1, \quad q = (v^2 - u^2)T + 1, \quad r = (v^2 + u^2)T + 1.$$

For the quantities p, q, r defined above we have

$$(p^2 - 1)^2 + (q^2 - 1)^2 - (r^2 - 1)^2 =$$
$$- 8T^3 u^2 (u + v)^2 (u^2 - 2uv + 3v^2) - 8T^4 u^2 (u + v)^2 (u^2 v^2 - 2uv^3 + v^4).$$

The polynomial on the right side of the above equality has two rational roots: $T = 0$ and

$$T = \frac{-u^2 + 2uv - 3v^2}{(u - v)^2 v^2}.$$

Using now the quantity T, definition of p, q, r and remembering that $t_{-x} = t_{x-1}$ we find rational parametric solution of the equation $t_x^2 + t_y^2 = t_z^2$ in the form

$$x(u, v) = \frac{u(u^2 - 2uv + 3v^2)}{(u - v)^2 v},$$

$$y(u, v) = \frac{(u + v)(u^2 - 2uv + 3v^2)}{2(u - v)v^2},$$

$$z(u, v) = \frac{u^4 - 2u^3 v + 2u^2 v^2 + 2uv^3 + v^4}{2(u - v)^2 v^2}.$$

Let us note that the solution we have obtained can be used in order to prove that our equation has infinitely many solutions in $\mathcal{S}$-integers, where $\mathcal{S}$ is finite set of primes and $2 \in \mathcal{S}$. Let us recall that we say that rational number r is $\mathcal{S}$-integer,

if the set of prime divisors of the denominator of r is contained in $\mathcal{S}$. Indeed, if we put $v = V^n, u = U^m - V^n$, where U, V are finite products of the elements from $\mathcal{S}$ and $2 \in \mathcal{S}$ we obtain rational numbers $x(u,v), y(u,v), z(u,v)$ which are $\mathcal{S}$-integers. The quantity of $\mathcal{S}$-integers solutions of our equation may suggest that the set of the integer solutions of the equation $t_x^2 + t_y^2 = t_z^2$ should be bigger (then 1).   □

Now we will consider the problem of construction of right angle triangles which legs are tetrahedral numbers. So, we are interested in the integer solutions of the diophantine equation

$$(3) \qquad\qquad z^2 = T_x^2 + T_y^2.$$

Let us note that $91^2 = T_5^2 + T_7^2$, so this equation has integer solution. W. Sierpiński in [4, page 57] wrote that it is unclear if the equation (3) has infinitely many solutions in integers. However, without much trouble we can construct infinitely many solutions ot this equation which satisfy the condition $y - x = 1$. Indeed, we have $T_{6x}^2 + T_{6x+1}^2 = (3x+1)^2(6x+1)^2 f(x)$, where $f(x) = 8x^2 + 4x + 1$. Because $f(0) = 1$ and the following identity holds

$$f(17x + 6z + 4) - (48x + 17z + 12)^2 = f(x) - z^2,$$

we can see that the equation $f(x) = z^2$ has infinitely many solutions x_n, z_n given by

$$x_0 = 0, \quad z_0 = 1, \quad x_n = 17x_{n-1} + 6z_{n-1} + 4, \quad z_n = 48x_{n-1} + 17z_{n-1} + 12.$$

From the above we can see that for each n we get the identity

$$((3x_n + 1)(6x_n + 1)z_n)^2 = T_{6x_n}^2 + T_{6x_n+1}^2.$$

In particular $T_{60}^2 + T_{61}^2 = 54839^2$, $T_{2088}^2 + T_{2089}^2 = 2150259925^2$, ...

In the light of the above result it is interesting to ask the question if it is possible to find infinite family of solutions x_n, y_n, z_n of the equation (3), with such a property that $y_n - x_n \to \infty$?

We will construct two families which satisfied mentioned condtion.

For the proof let us put

$$x(u,v) = v^2 - u^2 - 1,$$
$$y(u,v) = \frac{3v^2 - 2uv + 3u^2 - 3}{2},$$
$$z(u,v) = \frac{(v^2 - u^2)Z(u,v)}{192},$$

where

$$Z(u,v) = 105v^4 - 108uv^3 + (150u^2 - 96)v^2 - 4u(27u^2 - 16)v + 3(u^2 - 1)(35u^2 + 3).$$

For such x, y, z we get the following equality

$$T_x^2 + T_y^2 - z^2 = \frac{h(u,v)(h(u,v) + 2)H(u,v)}{36864},$$

where $h(u,v) = -1 + u^2 - 6uv + v^2$ and $H(u,v)$ is the polynomial of degree 8. Let us note that the equation $h(u,v) = 0$ has infinitely many solutions in positive integers. This is an immediate consequence of the equality $h(6, 35) = 0$ and the identity

$$h(u,v) = h(v, 6v - u).$$

This means that for the sequence defined recursively by the equations

$$u_0 = 6, \quad u_1 = 35, \quad u_n = 6u_{n-1} - u_{n-2},$$

we have equality $h(x_{n-1}, x_n) = 0$ for $n = 1, 2, \ldots$. We conclude that the numbers $x(u_{n-1}, u_n), y(u_{n-1}, u_n), z(u_{n-1}, u_n)$ are integer solutions of the diophantine equation $T_x^2 + T_y^2 = z^2$. In particular

$$T_{1188}^2 + T_{1680}^2 = 839790700^2, \quad T_{40390}^2 + T_{57120}^2 = 32946833683400^2, \ldots, .$$

Let us note that in order to prove above result we can also take

$$x'(u, v) = x(u, v),$$
$$y'(u, v) = y(u, v) + 1,$$
$$z'(u, v) = \frac{(v^2 - u^2)Z'(u, v)}{192},$$

where

$$Z'(u, v) = 105v^4 - 108uv^3 + (150u^2 + 96)v^2 - 4u(27u^2 + 16)v + 3(u^2 + 1)(35u^2 - 3).$$

For x', y', z' defined in this way we have an identity

$$T_{x'}^2 + T_{y'}^2 - z'^2 = \frac{h(u, v)(h(u, v) + 2)H'(u, v)}{36864},$$

where $h(u, v)$ is the same polynomial we have obtained previously and H' is a certain polynomial of degree 8.

We have proved

Theorem 2.4. (1) *The equation $T_x^2 + T_y^2 = z^2$ has infinitely many integer solutions satisfied the condition $y - x = 1$.*
(2) *There exists an infinite sequence (x_n, y_n, z_n) of solutions of the equation $T_x^2 + T_y^2 = z^2$ with such a property that $y_n - x_n \to \infty$.*

It is easy to see that the solutions of the equation (3) we have obtained are not co-prime. This suggest the following:

Question 2.5. *Does the equation $z^2 = T_x^2 + T_y^2$ have infinitely many solutions in integers x, y, z which satisfy the condition $\mathrm{GCD}(T_x, T_y) = 1$?*

In the range $x < y < 5 \cdot 10^4$ there are exactly 39 solutions of our equation and only one given by

$$x = 143, \quad y = 237, \quad z = 2301289,$$

satisfied the condition $\mathrm{GCD}(T_x, T_y) = 1$.
Unfortunately, we are unable to give an answer to the following

Question 2.6. *Does the equation $T_x^2 + T_y^2 = T_z^2$ have infinitely many solutions in rational numbers?*

3. Triangular numbers and palindromic numbers

Let us state the following

Question 3.1. *Let us fix $b \in \mathbb{N}_{>1}$. Is the set of triangular numbers which are palindromic in base b infinite?*

Let us remind that we say that the number k is palindromic in base b if in the system with base b

$$k = \sum_{i=0}^{m} a_i b^i$$

we have $a_i = a_{m-i}$ for $i = 0, 1, \ldots, m$.

In connection with this question we can prove the following

Theorem 3.2. *If $b = 2, 3, 5, 7, 9$, then there are infinitely many triangular numbers which are palindromic in base b.*

Proof. If $b = 2$, then for $n = 2^{2^k} + 1$ we have

$$t_n = (2^{2^k} + 1)(2^{2^k - 1} + 1) = 2^{2^{k+1} - 1} + 2^{2^k} + 2^{2^k - 1} + 1 = 1\underbrace{00\ldots00}_{k-zeros}11\underbrace{00\ldots00}_{k-zeros}1_2,$$

which proves that the number t_k is palindromic in base 2.

If $b = 3$ then we define $n = (3^k - 1)/2$ and we get number

$$t_n = \frac{3^{2k} - 1}{3^2 - 1} = 3^{2k-2} + 3^{2k-4} + \ldots 3^2 + 3^0,$$

which is clearly palindromic in base 3. Let us note that the number t_n is palindromic in base $b = 9$ due to the identity

$$t_n = \frac{9^k - 1}{9 - 1} = 11\ldots11_9.$$

In the case when $b = 5$ we put $n = (5^k - 1)/2$ and we get the number

$$t_n = \frac{5^{2k} - 1}{8} = 3 \cdot 5^{2k-2} + 3 \cdot 5^{2k-4} + \ldots + 3 \cdot 5^2 + 3 \cdot 5^0,$$

which is palindromic in base 5.

If now $b = 7$ then we define $n = (7^k - 1)/2$ and we get the number

$$t_n = \frac{7^{2k} - 1}{8} = 6 \cdot 7^{2k-2} + 6 \cdot 7^{2k-4} + \ldots + 6 \cdot 7^2 + 6 \cdot 7^0,$$

which is palindromic number in base 7. $\qquad\square$

Remark 3.3. Unfortunately, we are unable to prove that the set of palindromic triangular numbers in base 10 is infinite. Let us note that in the range $n < 10^6$ there are exactly 35 values of n with such a property that the number t_n is palindromic. However it is easy to see that there are infinitely many triangular numbers which are "almost" palindromic. This mean that at least one of the numbers $t_n \pm 1$ is palindromic. More precisely, if $n = 2 \cdot 10^{k+1} + 1$ then

$$t_n + 1 = 2\underbrace{00\ldots00}_{k-zeros}3\underbrace{00\ldots00}_{k-zeros}2.$$

If we take $n = 2 \cdot 10^k + 2$ then we have

$$t_n - 1 = 2\underbrace{00\ldots00}_{k-1-zeros}5\underbrace{00\ldots00}_{k-1-zeros}2.$$

4. Arithmetic progressions

In this section we consider the problem of construction of integer solutions of certain diophantine equations connected with values of some functions involving triangular and tetrahedral numbers in arithmetic progressions.

We start with the following

Theorem 4.1. *The equation*

$$\frac{1}{t_x} + \frac{1}{t_y} = \frac{2}{t_z}$$

has infinitely many non-trivial solutions in positive integers x, y, z. In other words: there are infinitely many three term arithmetic progressions consisted of the numbers $1/t_x, 1/t_z, 1/t_y$.

Proof. In order to prove our theorem let us put $z = (y - x - 1)/2$. Then we have an equality

$$t_{(y-x-1)/2} - \frac{2 t_x t_y}{t_x + t_y} = \frac{(x + y + 1)^2 f(x, y)}{8(x^2 + y^2 + x + y)},$$

where $f(x, y) = -x + x^2 - y - 4xy + y^2$. Note that $f(1, 5) = 0$ and that the following identity holds:

$$f(x, y) = f(y, 4y - x + 1).$$

Form the above we can deduce that if we define: $x_0 = 1$, $x_1 = 5$, $x_n = 4x_{n-2} - x_{n-2} + 1$, then for each n we have $f(x_n, x_{n+1}) = 0$ and additionally, if $n \equiv 1 \pmod 2$, then the number $(x_{n+1} - x_n - 1)/2$ is integer. This conclusion finishes the proof of our theorem.

In particular we have

$$\frac{1}{t_{76}} + \frac{1}{t_{285}} = \frac{2}{t_{104}}, \quad \frac{1}{t_{1065}} + \frac{1}{t_{3976}} = \frac{2}{t_{1455}}, \quad \frac{1}{t_{14840}} + \frac{1}{t_{55385}} = \frac{2}{t_{20272}} \ \dots \ .$$

$\square$

Theorem 4.2. *The diophantine equation $z^2 = (T_x + T_y)/2$ has infinitely many non-trivial solutions in integers x, y, z. In other words: there are infinitely many three term arithmetic progressions consisted of the numbers T_x, z^2, T_y.*

Proof. Proof of our theorem is an immediate consequence of the identity

$$\frac{T_{(u^2-1)/3} + T_{(2u^2-5)/3}}{2} = T_{u-1}^2$$

and the fact that for $u \equiv 1, 2 \pmod 3$ the values of the polynomials $(u^2-1)/3$, $(2u^2-5)/3$ are integers.

Let us note that we proved something more. Indeed, we have proved that there are infinitely many three term arithmetic progressions consisted of the numbers T_x, T_u^2, T_y. $\square$

Theorem 4.3. *The diophantine equation $t_z = \frac{x^4 + y^4}{2}$ has infinitely many non-trivial solutions in integers x, y, z. In other words: there are infinitely many three term arithmetic progressions consisted of the numbers x^4, t_z, y^4.*

Proof. By a non-trivial solution of the equation $t_z = \frac{x^4 + y^4}{2}$ we mean the solution x, y, z which is not of the form $x = m, y = m^2, z = m^4$ for certain $m \in \mathbb{N}$.

In order to prove our theorem it is enough to show that the diophantine equation
$(\star)$ $u^2 = 4x^4 + 4y^4 + 1$ has infinitely many solutions in integers. This is an easy
consequence of the fact that each triple of integers (x, y, u) which is a solution of
the equation $(\star)$ give us a triple of integers $(x, y, (u - 1)/2)$ which is a solution of
the equation $t_z = (x^4 + y^4)/2$.

Let us note the following identity

$$(130w^2 - 128w + 33)^2 = 4(60w^2 - 61w + 16)^2 + 4(5w - 2)^4 + 1.$$

This identity shows that the set of three terms arithmetic progression consisted of
the numbers x^2, t_z, y^4 is infinite. Thus, we can see that our theorem will be proved
if we were able to prove that the diophantine equation $v^2 = 60w^2 - 61w + 16 =: f(w)$
has infinitely many solutions in integers. Let us note that $f(0) = 4^2$, and next

$$(1921v + 14880w - 7564)^2 - f(248v + 1921w - 976) = v^2 - f(w).$$

Form the above identity we can deduce that if we define the sequences v_n, w_n
recursively by the equations

$$\begin{cases} w_0 = 0, \quad v_0 = 4, \\ w_n = 248v_{n-1} + 1921w_{n-1} - 976, \\ v_n = 1921v_{n-1} + 14880w_{n-1} - 7564, \end{cases}$$

then for each $n \in \mathbb{N}$ we have the identity $v_n^2 = f(w_n)$. This means that the equation
$t_z = (x^4 + y^4)/2$ has infinitely many solutions x_n, y_n, z_n given by

$$x_n = v_n, \quad y_n = 60w_n^2 - 61w_n + 16, \quad z_n = 65w_n^2 - 64w_n + 16, \qquad n = 0, 1, 2, \dots .$$

In particular we have $t_{16} = (4^4 + 2^4)/2$, $t_{15632} = (120^4 + 78^4)/2, \dots .$ $\qquad \square$

In the light of the above theorem it is natural to state the following

Question 4.4. *Does the equation $t_z = x^4 + y^4$ have infinitely many solutions in
integers?*

In the range $x < y < 10^5$ there are two solutions of this equation. There are the
following triples: $x = 15, y = 28, z = 1153$; $x = 3300, y = 7712, z = 85508608$.

5. VARIETES

We start with the following

Theorem 5.1. *There are infinitely many triangular numbers which are quotients
of tetrahedral numbers.*

Proof. It is easy to check that if

$$x(u) = u, \quad y(u) = \frac{u^3 + u^2 + 2u - 4}{2}, \quad z(u) = y(u)$$

or

$$x(u) = u, \quad y(u) = 3T_u, \quad z(u) = \frac{u^3 + u^2 + 2u + 2}{2},$$

then we have

$$t_{z(u)} = \frac{T_{y(u)}}{T_{x(u)}}.$$

This ends proof of our theorem. $\qquad \square$

In the light of the above theorem it is natural to ask about triangular numbers which are products of tetrahedral numbers. We can prove the following

Theorem 5.2. *There are infinitely many triangular numbers which are product of two tetrahedral numbers.*

Proof. We are interested in the integers x, y, z which satisfy the equation $t_z = T_x T_y$. In the table below we can find integer valued polynomials x_i, y_i, z_i which satisfy the equation $t_{z_i} = T_{x_i} T_{y_i}$ for $i = 1, \ldots, 9$.

$x_1(u)$	$9u$
$y_1(u)$	$(81u^3 + 27u^2 + 2u - 2)/2 =: f(u)$
$z_1(u)$	$(f(u) + 2)f(u)$
$x_2(u)$	$9u$
$y_2(u)$	$4f(u) + 1$
$z_2(u)$	$u(9u + 1)(9u + 2)(162u^3 + 54u^2 + 4u - 3)$
$x_3(u)$	$9u$
$y_3(u)$	$4f(u) + 5$
$z_3(u)$	$u(9u + 1)(9u + 2)(162u^3 + 54u^2 + 4u + 3)$
$x_4(u)$	$9u - 1$
$y_4(u)$	$(81u^3 - u - 2)/2 =: g(u)$
$z_4(u)$	$g(u)(g(u) + 2)$
$x_5(u)$	$9u - 1$
$y_5(u)$	$4g(u) + 1$
$z_5(u)$	$u(9u - 1)(9u + 1)(162u^3 - 2u - 3)$
$x_6(u)$	$9u - 1$
$y_6(u)$	$4g(u) + 5$
$z_6(u)$	$u(9u - 1)(9u + 1)(162u^3 - 2u + 3)$
$x_7(u)$	$9u - 2$
$y_7(u)$	$(81u^3 - 27u^2 + 2u - 2)/2 =: h(u)$
$z_7(u)$	$h(u)(h(u) + 2)$
$x_8(u)$	$9u - 2$
$y_8(u)$	$4h(u) + 1$
$z_8(u)$	$u(9u - 2)(9u - 1)(162u^3 - 54u^2 + 4u - 3)$
$x_9(u)$	$9u - 2$
$y_9(u)$	$4h(u) + 5$
$z_9(u)$	$u(9u - 2)(9u - 1)(162u^3 - 54u^2 + 4u + 3)$

$\square$

Theorem 5.3. *The diophantine equation $t_p^2 + t_q^2 = t_r^2 + t_s^2$ has infinitely many non-trivial solutions in integers.*

Proof. In order to prove our theorem it is enough to show that the diophantine equation $(x^2 - 1)^2 + (y^2 - 1)^2 = (u^2 - 1)^2 + (v^2 - 1)^2$ has infinitely many solutions in odd integers. We use the method which is very close to the method employed by Euler during his investigation of integer solutions of the diophantine equation $p^4 + q^4 = r^4 + s^4$, [2, page 90].

Let us define $f(x, y) = (x^2 - 1)^2 + (y^2 - 1)^2$ and note that if

$$x = T + c, \quad y = bT - d, \quad u = T + d, \quad v = bT + c,$$

then $f(x, y) - f(u, v) = -2a_1T - 6a_2T^2 - 4a_3T^3 =: g(T)$, where

$$a_1 = (b-1)c(c^2 - 1) - (b+1)d(d^2 - 1),$$
$$a_2 = (b^2 - 1)(c^2 - d^2),$$
$$a_3 = c(b^3 - 1) + d(b^3 + 1).$$

If we put $c = -b^3 - 1$, $d = b^3 - 1$, then $a_3 = 0$ and the equation $g(T) = -8b^3(b^2 - 1)T(3T + b^4 - 2b^2 - 2) = 0$ has two rational roots: $T = 0$ and

$$T = -(b^4 - 2b^2 - 2)/3.$$

Using the values of T we have found and remembered that $t_{-n} = t_{n-1}$ we can find solutions of the equation from the statement of our theorem in the form

$$p(b) = \frac{(b+1)(b^3 + 4b^2 + 2b + 2)}{6}, \quad q(b) = \frac{b^5 + b^3 - 2b - 6}{6}$$
$$r(b) = \frac{(b+1)(b^3 - 4b^2 + 2b - 2)}{6}, \quad s(b) = \frac{b(b^2 - 1)(b^2 + 2)}{6}.$$

It is easy to see that if $b \equiv 1 \pmod 3$, then the numbers $p(b), q(b), r(b), s(b)$ are integers. $\square$

W. Sierpiński in the paper [7] proved that the equation $z^2 = T_x + T_y$ has infinitely many solutions in integers. Next step is the question if similar result can be proved if a cube instead of a square is considered. As we will see the answer on such modified question is affirmative

Theorem 5.4. *There are infinitely many cubes which are sums of two tetrahedral numbers.*

Proof. Let us note the following equality

$$\left(\frac{x + 6y}{2}\right)^3 - T_{x+5y-1} - T_{y-1} = \frac{1}{24}(x + 6y)F(x, y),$$

where $F(x, y) = x^2 - 24y^2 - 4$. In order to finish the proof we must show that the diophantine equation $F(x, y) = 0$ has infinitely many solutions in positive integers. In order to prove this let us note that $F(2, 0) = 0$ and next that

$$F(5x + 24y, x + 5y) = F(x, y).$$

Thus we can see that if we define

$$x_0 = 2, \quad y_0 = 0, \quad x_n = 5x_{n-1} + 24y_{n-1}, \quad y_n = x_{n-1} + 5y_{n-1},$$

then for each $n \in \mathbb{N}$ we have $2|x_n$ and $F(x_n, y_n) = 0$. This shows that the equation $z^3 = T_x + T_y$ has infinitely many solutions in integers. In particular we have: $T_1 + T_{19} = 11^3$, $T_{19} + T_{197} = 109^3$, $T_{197} + T_{1959} = 1079^3 \ldots$. $\square$

In the light of the result of Sierpiński and the above theorem it is natural to ask the following

Question 5.5. *For which n the diophantine equation $z^n = T_x + T_y$ has a solution in positive integers?*

If $n = 4$ then the smallest solution of this equation is: $x = 8$, $y = 38$, $z = 10$. Let us note that in the range $x < y < 10^4$ this equation has exactly six integer solutions.

Theorem 5.6. *There are infinitely many pairs of different tetrahedral numbers with such a property that their product is a square of integer.*

Proof. We consider the problem of the existence of integer solutions of the diophantine equation $z^2 = T_x T_y$. In order to prove our theorem it is enough to show that the equation $v^2 = (x+2)(2x+1)/9 = f_1(x)$ has infinitely many solutions in integers. Indeed, this is simple consequence of the identity $T_x T_{2x} = x^2(x+1)^2 f_1(x)$. We can easily check the identity

$$(8u + 17v + 10)^2 - f_1(17u + 36v + 20) = v^2 - f_1(u).$$

Because $f_1(1) = 1$ we can see that if we define sequences u_n, v_n recursively in the following way

$$u_0 = 1, \quad v_0 = 1, \quad u_n = 17u_{n-1} + 36v_{n-1} + 20, \quad v_n = 8u_{n-1} + 17v_{n-1} + 10,$$

then for each $n \in \mathbb{N}$ the following identity holds

$$(v_n u_n (u_n + 1))^2 = T_{u_n} T_{2u_n}.$$

In particular $189070^2 = T_{73} T_{146}$, $7559616818^2 = T_{2521} T_{5042}$, ...

At the end let us note that the proof of our theorem can be performed with the use of the identity $T_x T_{2x+2} = (x+1)^2(x+2)^2 f_2(x)$, where $f_2(x) = x(2x+3)/9$. It is easy to prove that the equation $v^2 = f_2(u)$ has infinitely many solutions in integers.

In the light of the proof of our theorem and the remark above we can state an interesting question concerning the existence of infinite set of triples of integers x, y, z which satisfy the equation $z^2 = T_x T_y$ with the condition $2x + 2 < y$? $\square$

Remark 5.7. We do not know, if there exist three different tetrahedral numbers in geometric progression, but it is possible to prove that there are infinitely non-trivial rational solutions of the diophantine equation $T_x T_y = T_z^2$. Proof of this fact can be found in [10].

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